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On Application of Fixed Point Theorem to Insurance Loss Model

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Abstract

The future development when an insurance company is in a difficult circumstance can be described by a stochastic process which the insurance company is tasked to manage effectively in order to achieve best goal of the company. Application of an effective risk or loss management model in an insurance company brings in more revenue for the insurer and less conditional pay-out of claims to the insured. Insurance losses, risks and premium calculation or pricing have been active and essential topics in insurance and actuarial literatures but most of these literatures did not only stand the test of time due to dynamic nature of insurance principles and practices in highly evolving environment but also lack the intuitive and detailed standard rating logic to adjust loss rating to a particular experience. There is a need to strike a balance in charging an appropriate and equitable premium by applying a suitable loss model that gives a sufficient uniquely determined solution that will not necessarily put an insurer or the insured in uncertain awkward business situations. Therefore, the objective of this research is to obtain sufficient conditions for convergence of algorithm towards a fixed point under typical insurance loss and actuarial circumstances to achieve a uniquely determined solution. At the end, a unique fixed point was determined and the algorithm formulated converges towards that point through straightforward and simplified generalised formulae and functions.

Keywords: Fixed point theorem; uniqueness; insurance loss; convergence; risk variables.

1. Background to the Study.

The future development when an insurance company is in a difficult circumstance can be described by a stochastic process which the insurance company is tasked to manage effectively in order to achieve best goal of the company. Fixed point theorem is very general but we shall narrow its application to insurance business problem to get precise formulation. In studying some nonlinear phenomena, fixed point theorem is an important and powerful tool that can be applied in many fields. This research paper attempts to apply fixed point theorem in the areas of insurance business just as it has been applied to geometry, analysis, number theory, set theory, group theory, algebra, dynamics, topology and so on. It is noteworthy to explain briefly what a fixed point theory means. According to (Rajic, Azdejkovic, & Loncar, 2014), fixed point theorem concerns itself with the examination of the existence of a certain point, say y, in the domain of a function, say g, where g(y)=y. The identical function mapping and the function values are equal. This means that any marginal change in the function of y will proportionally result to additional fixed points. If g(y) = y then g(y) - y =0. Therefore if a certain function f is shown as f(y) = g(y) – y, function g has zero as the fixed point. If Y is a set and $g:Y \rightarrow Y$ is a map from Y to Y, a point $y \in Y$ is known as a fixed point of g since g(y) = y. For a family of G of Y, G is a semigroup or group. Here the fixed point theorem gives specific condition on Y and G ensures that there exists a simultaneous fixed point in $y \in Y$ for $g \in G$. If G is a group F, it arises from a group action Ω:F*Y→Y of F on Y. If Ω(F, Y) := fy, it is assumed that 1 of F is the identity of Y. Also, if $a \in F$ and $y \in Y$ for all f such that (fa)y = f(ay) such y = F-space, Y = topological space, F = topological ground and Ω is jointly continuous. If there exists an action group F on x where Y(power set)=P(x), it leads to an action of F on P(x) and the F-invariant (subset of X) is the fixed point of the new action. In other words, the fixed point theorem leads to an F-invariant set. Under the existence of Haar measure Ψ on a compact topological group F, left invariance indicates that Ψ is F- fixed due to the action of F by left translation on the space N^{+}(F) of any finite positive measure on F. If Ψ is normalised and finite so that Ψ(F)= 1, it can be referred to as F-fixed point under the action of F on the convex set of normalised measure on F

The research work of (Fu & Wu, 2005) concentrated extensively on the iterations of loss rating algorithms while in the research work of (Borogovac, 2014), the clear concept of Loss Ratio Method (also known as the Standard Rating Method) was introduced. All these and many other similar papers lack the intuitive and detailed standard rating logic to adjust loss rating to a particular experience. The objective of this research is to obtain sufficient conditions for convergence of algorithm towards a fixed point under typical insurance loss and actuarial circumstances to achieve a uniquely determined solution. This paper will address the shortcoming of the actuarial algorithms sighted in a concise manner. The paper will be significant and serves as a solid foundation for further research on the application of fixed point theorem in insurance premium calculation, pricing or fixing because the demand for insurance products is majorly a function of the cost of insurance and the premium charged by insurers.

2. Brief Review.

Fixed point theories came to limelight in Mathematics in the 19th century. Henry Poincare applied them in nonlinear problem topological analysis. After Poincare in 1886, L.E.J (Bertus) Brouwer also contributed to the development of fixed point theorem in 1912 according to (Khalehoghli, Rahimi, & Gordji, 2020). (Istratescu, 1981) shed more light on how Brouwer's fixed point theorem gave better understanding on differential equations worked on by Charles Emile Picard and Henri Poincare. Brouwer was the first to prove that the fixed point theorem for the function of x is x (that is, f(x) = x). It follows that any continuous function g mapping compact convex set to itself, there exists a point y_0 such that $g(y_0) = y_0$. For a continuous function g from a closed interval c_i in the real numbers from a closed disk c_d to itself, there exists at least a fixed point. Also, every continuous function in a Euclidean space from a closed ball into itself must equally have a fixed point. This theorem holds for endomorphic functions which have the same set as range and the domain. But consider a function g(y) = y+1 with domain [-1, 1] and the range [0, 2]. Here function g is not an endomorphism because if the same continuous function from $\mathbb{R}$ to itself is considered, it will have no fixed point as it shifts to the right. The fixed point theorem proved by Brouwer became an eye opener and widely used by many people as it is pervasive and diverse in mathematical applications. The theorem pioneered and broadened a number/degree of generalisations to different aspects of scientific disciplines. (Von-Neumann, 1928), in his work, used Brouwer's theorem to prove the existence of balanced growth equilibrium and minimax solution to expanding economy and two-agent game respectively. In the same vein, Kakutani in 1941 further developed and extended the work of Brouwer to prove the fixed point theorem for a sphere, square and their equivalent n-dimensional counterparts. According to (Nash, 1951), the work of Kakutani in 1941 made it simple to use fixed point theorem to prove the complex theorem in non-cooperative games. (Nieto & Guez-Lo'pez, 2005) showed how Banach, in 1922, demonstrated that contractive mapping with complete domain possesses a unique and peculiar fixed point. The Banach Principle was latter extended by Nadler in 1969 and applied to set valued mapping in metric space.

Insurance losses, risks and premium calculation, pricing or fixing have been active and essential topics in insurance and actuarial literatures as examined by Goovaert, De-Vylder and (Goovaerts, De Vylder, & Haezendonck, 1984). In their work, they examined insurance premium and its theory with application. This work did not stand the test of time due to dynamic nature of insurance principles and practices in highly evolving environment. (Hurlimann, 1997) researched on quasi-mean value principles while mathematics methods in risk theory was worked on by (Buhmann, 1970) but situations of things are different now as most of these theories and principles are outdated or inefficient now, although they were exceptional and outstanding at that time interval. In the same vein, the insurance premium calculation in relation to modern theories and risk choices under uncertainty have been studied in the work of (Wang, Young, & Panjer, 1997); (Hurlimann, 1998); (Wang & Young, 1998) and so on. Most of these studies only emphasised on reasonable and desirable properties and characteristics insurance premium should possess and satisfy. In other words, they were based on expected utility principles and dual theory. According to (Mallappa & Talawar, 2020), the P_{max} (maximum premium or consideration that a particular insured is willing to give the insurer) can best be determined by applying the concepts of utility theory by looking into different utility functions following the assumptions of insurance random variables to apply. For actuaries, one of the most important objectives in insurance industry is how to apply suitable principles and models for premium calculation because the premium reflects loading(expenses), selection and certain expected estimated claims(Buhlmann, 1970). An insurer sets different premiums for different risks attached to different insurance policies. This is necessary in an insurance contract in order for it to be mutually advantageous and fair to both the insured and the insurance company because premium calculation is individually and differently affected by mortality, interest rate, loading (expenses), age, sex, utility function, moral hazard and so on.

3. Input-Output System of Insurance Activities.

3.1 Insurance as a Dynamic System

Insurance companies generate their revenue majorly from the premium they charge the insured for insurance coverage and from the effective/efficient investment of the premium to yield returns. The premium charged by insurers can be placed into savings account for safety of funds but that will not be totally advisable because of exposure of funds to inflationary risk. Alternatively, insurance premium can be invested in short-term assets or investment such as treasury bills, interest-bearing cash equivalents, high-grade cooperate bonds and so on. Application of an effective risk or loss management model in an insurance company brings in more revenue for the insurer and less conditional pay-outs of claims to the insured. The premium and other revenue of insurance company represent the input while the output represents the claim handling costs and other insurance costs incurred by the insurer. Therefore, it will not be wrong to state that the insurance activities follow input-output systems as shown in Figure 1.

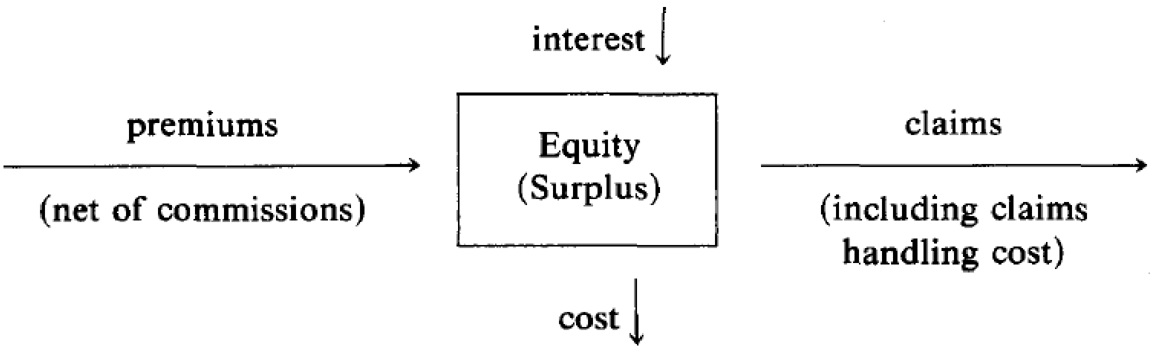

Figure 1: Insurance as a Dynamic System

Underwriting is an important concept in premium calculation. The insurer will charge the insured inappropriately (more or less) for a particular risk if good underwriting skill is not applied. In other words, without good underwriting practice, the insurer may undercharge an insured with under-average or substandard risk while overcharging the less risky situations.

3.2 Insurance Premium Model

The insurance company business models are based around various assumptions and diversifications of risks which entail pooling of insured risks and spreading of these risks across larger portfolios. Therefore, a qualified actuary in an insurance company must make use of an appropriate insurance loss model to price a risk, otherwise the insurer will likely be in a ruin. For instance, if too much hurricane insurance is written using a model that assumes no or low chance of hurricane inflicting environment, such insurance company may be pushed into a ruin if unimaginable havoc wrecking hurricane occurs. That is, the insurer company may be pushed out of business if the hurricane hits the insured region. Let's assume a market for motor insurance where the insured are identical except for the time and the probabilities of loss (accident) occurrence that differ. If there are L insured where L = 1, 2, …, m.

$$L = \sum_{i=1}^{m} i$$

The insured's (L's) probability of loss is denoted by $P_L \in [0, 1]$ as the occurrence of losses is independent among the insured. If an insured with a wealth premium of m is involved in an accident (loss) worth N naira with Y naira full cost of insurance and purchased by L insured, then the insurer will expect a profit of ₦$(Y - P_L N)$. If ₦Y_L is the cost of insurance policy paid to L insured who suffered losses, the goal of the insurer would be to determine for each L=1,2,…, m the appropriate price using a suitable model for insurance policy L because if Y_L is less than $P_L N$, selling such policy will result to a loss , thereby, leading the insurance supply of the policy to be equalled to zero. In other words, the supply of such policy will be infinite or desirable if the Y_L is greater than the P_L N. It will be a breakeven on each policy L at the point where $P_L N = Y_L$.

It will be interesting to note that when cost of insurance increases drastically, it will bring more profit for insurer but the expected utility an insured gets from purchasing the policy will decrease in that same manner in most cases because it will appear to the proposed insured that it will be of no or low use to procure such insurance policy when compared to other loss prevention techniques. The remaining insured who will continue to buy the insurance policy at any cost are those who have no other loss prevention options or those who their estimated cost of not insuring at that premium is way higher than when insured. In a nutshell, at an increased cost of procuring insurance policy, it becomes riskier for the insurer because of the composition of the pool of the insured at that time. It must also be noted that just as it is riskier for increased cost, it is absolutely riskier when an insurer is undercharging the insured in order to get high patronage. Apart from the insurer risking a situation of not meeting the claim demand of the insured which can dent the image or goodwill of such insurer, it will also have a greater adverse/negative effect on the profit maximisation goal of the insurance company which will in turn outweighs the positive influence of the patronage benefits. Therefore, there is a need to strike a balance in charging an appropriate and equitable premium by applying a suitable loss model that gives a sufficient uniquely determined solution that will not necessarily put an insurer in an uncertain awkward business situations. This research will employ fixed point theorem to achieve this aim.

4. Application of Fixed Point Theorem in Insurance.

4.1 preliminaries

Fixed point theorem has been applied in different aspects or areas of sciences, especially in engineering. It has been used extensively in Mathematics to solve equations, simulate and approximate especially in game theory. It is also used in economics to determine and investigate the point of equilibrium in demand and supply functions. It has helped many economists to understand complex problems in the generic economic models like computation/stability, comparative statics, and robustness of marginal change.

Let us consider 2 vector random variables p and q with a and b categories;

$$\sum_{j=1}^{b} c_{ij}\left(d_{ij} - p_i q_j\right) = 0, \qquad i = 1,2,\dots,a$$

$$\sum_{i=1}^{a} c_{ij}\left(d_{ij} - p_i q_j\right) = 0, \qquad j = 1,2,\dots,b$$

Where: d_{ij} = observed loss costs

c_{ij} = earned risk exposure

The relatives p_i and q_j are unknown as the system contains a+b equations. Let us consider a vector equation;

$$\Omega(g) = 0 \qquad (4.1)$$

Where $\Omega : D_\Omega \to \mathbb{R}^r$, integer $r \in N$ represents generic category of Euclidean space. The domain of Ω is $D_\Omega \subseteq \mathbb{R}^r$. From this, the corresponding vector equation is derived;

$$g = \beta(g) \qquad (4.2)$$

where $D_\beta \to \mathbb{R}^r$ and the domain of β is $D_\beta \subseteq \mathbb{R}^r$. This leads to the formula of the corresponding algorithms;

$$g^{w+1} := \beta(g^w), \text{ where } w= 0,1,2,\dots \qquad (4.3)$$

The iteration formula in the form of (4.3) can be derived directly from the standard loss ratio method adopted by (Borogovac, 2014). Also, the main characteristics of mathematically formalised rating algorithm model can be reiterated for simplicity amd determination of sufficient conditions for convergence of the algorithm towards a fixed point under typical actuarial circumstances. Let us first examine the concept of functional analysis.

4.2 Functional Analysis

For $g = (g_1, \dots, g_r) \in \mathbb{R}^r$, the functions defined by:

$$\|g\|_x := (f_1^x + \cdots + f_r^x)^{1/x}, \text{ for } x > 1$$

$$\|g\|_x := |g_1| + \dots + |g_r| \text{ and}$$

$$\|g\|_\infty := \max\{|g_1| + \dots + |g_r|\}$$

are considered norms in $\mathbb{R}^r$. Euclidean norm is derived when x=2. Thus $\|g\|_\infty \leq \|g\|_x \leq \|g\|_1$. Since all norms in $\mathbb{R}^r$ are the same, the convergence of the sequence $\{g^w\} \subseteq \mathbb{R}^r$

towards one norm means the same as the convergence towards $\bar{g}$ (same vector of another norms). The norm of a linear mapping $N : \mathbb{R}^b \rightarrow \mathbb{R}^a$ is defined by

$$\| N \| := \sup_{g \in R^r - \{0\}} \frac{\| Ng \|}{\| g \|}$$

In this research, we will concentrate on $\| \bullet \|_1$ and $\| \bullet \|_\infty$ since all norms in $\mathbb{R}^r$ have the same representation. If $: D_\beta \rightarrow \mathbb{R}^r$, $\forall D_\beta \subseteq \mathbb{R}^r$, is differentiable at $p \in D_\beta$, then

$J_\beta (p) = \frac{\alpha (\beta_1,\ldots,\beta_r)}{\alpha (p_1,\ldots,p_r)}$ represents the Jacobi Matrix at p.

Therefore,

$$J_\beta (p) = \begin{pmatrix} \frac{\alpha \, \beta_1 (p)}{\alpha \, p_1} & \cdots & \frac{\alpha \, \beta_1 (p)}{\alpha \, p_r} \\ \vdots & \ddots & \vdots \\ \frac{\alpha \, \beta_r (p)}{\alpha \, p_1} & \cdots & \frac{\alpha \, \beta_r (p)}{\alpha \, p_r} \end{pmatrix}$$

For matrix norms $\| \bullet \|_1$ and $\| \bullet \|_\infty$, we can derive :

$$\| J_\beta (p) \|_1 = \max_{j=1,\ldots,r} \{ \left| \frac{\alpha \, \beta_1 (p)}{\alpha \, p_j} \right| + \cdots + \left| \frac{\alpha \, \beta_r (p)}{\alpha \, p_j} \right| \} \quad (4.4)$$

and

$$\| J_\beta (p) \|_\infty = \max_{i=1,\ldots,r} \{ \left| \frac{\alpha \, \beta_i (p)}{\alpha \, p_1} \right| + \cdots + \left| \frac{\alpha \, \beta_i (p)}{\alpha \, p_r} \right| \} \quad (4.5)$$

Recall $U \subseteq \mathbb{R}^r$ denotes a convex set if $(g, f \in U, w \in [0, 1]) \rightarrow wg + (1 - w)f \in U$

4.3 Rating Model

The risk factors measure the risk space or risk cell set. Here we will consider three risk factors or classification variables which will be represented by vectors below;

$$p = (p_1, p_2, \ldots, p_a), \; q = (q_1, q_2, \ldots, q_b), \; r = (r_1, r_2, \ldots, r_c)$$

where p_i, q_j and r_k are not less than or equal to zero for

$$i=1,\ldots,a; \qquad j=1,\ldots,b \qquad \text{and } k=1, \ldots, c \qquad (4.6)$$

In this case, the set satisfying (4.6) is $D^o \subset \mathbb{R}^{a+b+c}$.

The risks (p_i, q_j and r_k) are assigned to cells i, j and k. We derive a plane in the risk space if we assume one index is fixed. The cell with the maximum risk exposure or statistical credibility is taggged the base cell. The model is multiplicative since the since the base rate (d_{111}) corresponds to the cell (1,1,1) . That is, the rate by the triplet risks (p_i, q_j and r_k) is calculated by;

$$d_{ijk} = d_{111} \, p_i q_j r_k \qquad (4.7)$$

From (4.7), $p_1=1$, $q_1=1$ and $r_1=1$. Hence $1= p_1=q_1=r_1$

If $H= (h_{ijk})_{apbpc}$, where $h_{ijk} \geq 0$, represents trended and fully developed insurance losses in Nigeria. h_{ijk} denotes projected or expected loss in ₦aira in the cell i,j,k. based on previous loss experience, the projected insurance losses h_{ijk} can be derived. The total projected or expected insurance loss can be expressed as;

$$H = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} h_{ijk}$$

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The corresponding earned risk exposures (L) can now be expressed as;

$$L = (l_{ijk})_{apbpc}\text{, where } l_{ijk} \geq 0$$

l_{ijk} represents the units of insurance in the cell (i, j and k) sold, as both H and L denote the insurance business past experience information using (4.7). Risk exposure serves as a necessary and expected positive condition in insurance practice. Therefore, without loss of generality,

$$h_{ijk} \geq 0 \rightarrow l_{ijk} \geq 0 \tag{4.8}$$

Rating model is important and necessary in actuarial application because it helps to adjust available rates to suit the information given by H and L. Intuitively, the adjusted risk or indicated factors or variables are represented by;

$\hat{p}=(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_a)$, $\hat{q}=(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_b)$ and $\hat{r}=(\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_c)$

The corresponding indicated rates are represented by;

$$\hat{d}_{ijk} = \hat{d}_{111}\, p_i q_j r_k \tag{4.9}$$

Summing loss amount, having kept fixed one index at a time, the vector of losses for factor p is denoted by h^p .

$$h_i^p = \sum_{j=1}^{b}\sum_{k=1}^{c} h_{ijk} \quad i = 1,2, \ldots, a \tag{4.10}$$

Also, h_j^q and h_k^r represent loss vectors for factors q and r respectively.

Invariably,

$$H=\sum_{i=1}^{a} h_i^p = \sum_{j=1}^{b} h_j^q = \sum_{k=1}^{c} h_k^r$$

The adjusted risk exposure of factor p can now be defined as;

$$L_i^p = \sum_{j=1}^{b}\sum_{k=1}^{c} l_{ijk} q_j r_k \quad i = 1,2, \ldots, a \tag{4.11}$$

Then the equivalent loss costs for factor p is represented by

$$H_i^p = \frac{h_i^p}{L_i^p}, \qquad i = 1,2,\ldots,a \tag{4.12}$$

This leads to the derivation of the following indicated factors;

$$\widehat{p_\imath} = \frac{H_i^p}{H_1^p}, \qquad i = 1,2,\ldots,a \tag{4.13}$$

$$\widehat{q_\jmath} = \frac{H_j^q}{H_1^q} j = 1,2,\ldots,b$$

$$\widehat{r_k} = \frac{H_k^r}{H_1^r}, \qquad k = 1,2,\ldots,c$$

The formulae calculated the indicated factors for $\hat{p}$, $\hat{q}$ and $\hat{r}$ using exposures and losses as inputs. This is the standard loss ratio method which has made actuarial work easier in calculating the indicated rates and factors without the need for insurance premiums. Previously, calculating indicated rates and factors without insurance premium value was nearly and practically impossible as confirmed by (Brown & Gottlieb, 2001). In the same vein, indicated base rate is denoted by;

$$\hat{d}_{111} = \frac{H}{PR} \frac{1}{\sum_{i=1}^{a} \sum_{j=1}^{b}\sum_{k=1}^{c} \hat{p}_i \hat{q}_j \hat{r}_k l_{ijk}} \tag{4.14}$$

Where PR= loss ratio permissible.

Comparing (4.9) and (4.14), indicated rates can be obtained in terms of indicated factors which depend only on loss and exposure formulae

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5. Risk Rating Algorithm and Application

5.1 Indicated Rates/Factors

Let us represent the current insurance risk factors by p^0, q^0 and r^0. The relativities after w^{th} iterations will be p^w, q^w and r^w. Using formula (4.13), p_i, q_j and r_k are w^{th} iteration values. If $\widehat{p_i}$, $\widehat{q_j}$ and $\widehat{r_k}$ are values of the next iteration which is $(w+1)^{th}$ iteration;

$$\widehat{p_i} = \frac{h_i^p \sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{1jk}}{h_1^p \sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{ijk}} \quad (5.1)$$

$$\widehat{q_j} = \frac{h_j^q \sum_{i=1}^{a} \sum_{k=1}^{c} p_i r_k l_{i1k}}{h_1^q \sum_{i=1}^{a} \sum_{k=1}^{c} p_i r_k l_{ijk}}$$

$$\widehat{r_k} = \frac{h_k^r \sum_{i=1}^{a} \sum_{j=1}^{b} p_i q_j l_{ij1}}{h_1^r \sum_{j=1}^{b} \sum_{k=1}^{c} p_i q_j l_{ijk}}$$

If the values of the coefficients h_i^p, h_j^q, h_k^r and l_{ijk} remain constant where ;

$i = 1, 2, \ldots, a$
$j = 1, 2, \ldots, b$
$k = 1, 2, \ldots, c$

we can continue to iterate repeatedly by substituting $\widehat{p_i}$, $\widehat{q_j}$ and $\widehat{r_k}$ for p_i, q_j and r_k into the formula (5.1) so as to make the iteration process sufficiently close as;

$\widehat{p_i} \approx p_i$, $i = 1,2,\ldots,a$
$\widehat{q_j} \approx q_j$, $j = 1,2,\ldots,b$
$\widehat{r_k} \approx r_k$, $k = 1,2,\ldots,c$

It is now easier to compute $\hat{\hat{d}}_{ijk}$ (the final indicated rate) upon the convergence of the iteration process by using (4.9), (4.13) and (4.14)

5.2 Uniqueness of Determined Solutions in Actuarial Practice.

To prove that our rating algorithm converges to a unique solution, Brouwer's Fixed Point Theorem will be recalled by restating the following:

Lemma 5.1: There exists a point $\bar{g}$ such that $\beta(\bar{g}) = \bar{g}$ for continuous function β mapping a convex and compact set U of an Euclidean space into itself.

It must be noted that under the condition of Brouwer's Theorem, there is no guarantee for the uniqueness of the fixed point $\bar{g}$.

Theorem 5.2: If arrays $H = (h_{ijk})_{apbpc}$ and $L= (l_{ijk})_{apbpc}$ are positive values satisfying the following:

(1) $h_1^p > 0$, $h_1^q > 0$ and $h_1^r > 0$;

(2) $l_{ijk} > 0$, where $i = 1,2,\ldots,a$; $j = 1,2,\ldots,b$; $k = 1,2,\ldots,c$

As defined in (5.1), it contains at least one solution $\bar{g} \in \mathbb{R}^{a+b+c}$ where $\mathbb{R}^{a+b+c}$, $\mathbb{R}^a$, $\mathbb{R}^b$ and $\mathbb{R}^c$ are all Euclidean spaces. The vector $g := (p; q; r) := (p_1,\ldots, p_a ; q_1,\ldots,q_b ; r_1,\ldots,r_c) \in \mathbb{R}^{a+b+c}$ and the function $\beta: \mathbb{R}^{a+b+c} \rightarrow \mathbb{R}^{a+b+c}$ can be introduced as:

$$\beta_i(\bar{g}) = \frac{h_i^p \sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{1jk}}{h_1^p \sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{ijk}}, \quad i=1,2,\ldots,a. \quad (5.2)$$

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$$\beta_{a+j}(\bar{g}) = \frac{h_j^q \sum_{i=1}^{a} \sum_{k=1}^{c} p_i r_k l_{i1k}}{h_1^q \sum_{i=1}^{a} \sum_{k=1}^{c} p_i r_k l_{ijk}}, \quad j = 1,2,\ldots,b$$

$$\beta_{a+b+k}(\bar{g}) = \frac{h_k^r \sum_{i=1}^{a} \sum_{j=1}^{b} p_i q_j l_{ij1}}{h_1^r \sum_{j=1}^{b} \sum_{k=1}^{c} p_i q_j l_{ijk}}, \quad k = 1,2,\ldots,c$$

where the denominators of β_i , β_{a+j} and $\beta_{a+b+k} \neq 0$ for the continuous vector functions.

Given that: $\theta_i^p := \min\{l_{ijk}: \ j = 1, \ldots, b\,; k = 1, \ldots, c\}$,

$N_i^p := \max\{l_{ijk}: \ j = 1, \ldots, b\,; k = 1, \ldots, c\}$, $i = 1, \ldots, a$;

$\theta_j^q := \min\{l_{ijk}: \ i = 1, \ldots, a\,; k = 1, \ldots, c\}$,

$N_j^q := \max\{l_{ijk}: \ i = 1, \ldots, a\,; k = 1, \ldots, c\}$, $j = 1, \ldots, b$;

$\theta_k^r := \min\{l_{ijk}: \ i = 1, \ldots, a\,; j = 1, \ldots, b\}$,

$N_k^r := \max\{l_{ijk}: \ i = 1, \ldots, a; \ j = 1, \ldots, b\}$, $k = 1, \ldots, c$

Without loss of generality, the second condition stated ealier is the same as θ_i^p , θ_j^q and θ_k^r where $i = 1, \ldots, a$; $j = 1, \ldots, b$ and $k = 1, \ldots, c$ respectively. Taking a closer look at this condition, the values of l_{ijk} must not be equal to zero in any circumstance otherwise the box U will not be bounded and $\beta(\bar{g})$ will not be a continuous function because if $l_{ijk} = 0$, it obviously means that $\theta_i^p = \theta_j^q = \theta_k^r = 0$. The implication is that the Brouwer's Fixed Point Theorem will be unfulfilled.

Invariably; $\frac{h_i^p \theta_1^p}{h_1^p N_i^p} \leq \beta_i \leq \frac{h_i^p N_1^p}{h_1^p \theta_i^p}$, $i = 1, \ldots, a.$ (5.3)

$\frac{h_j^q \theta_1^q}{h_1^q N_j^q} \leq \beta_{a+j} \leq \frac{h_j^q N_1^q}{h_1^q \theta_j^q}$, $j = 1, \ldots, b$

$\frac{h_k^r \theta_1^r}{h_1^r N_k^r} \leq \beta_{a+b+k} \leq \frac{h_k^r N_1^r}{h_1^r \theta_k^r}$, $k = 1, \ldots, c$

As defined by the inequalities in (5.3) ,

the vector $\{\beta_1, \ldots, \beta_a\,; \ \beta_{a+1}, \ldots, \beta_{a+b}\,; \ \beta_{a+b+1}, \ldots, \beta_{a+b+c}\} \sqsubseteq U \subseteq \mathbb{R}^{a+b+c}$.

Also, formula (5.2) maps initial guess $g^0 := (p^0; q^0; r^0)$ into U and maps U into itself by applying formula (5.3). Then, Box U (bounded and closed) is a convex and compact set in the Euclidean space with the continuous function β. Therefore, all conditions of Lemma 5.1 are now satisfied. Hence, by the function defined by (5.2), there exists a fixed point $\bar{g}$ such that $\beta(\bar{g}) = \bar{g}$. Therefore, the theorem is proved.

In order to sufficiently discuss the uniqueness of solution, let us further consider Lemma 5.3.

Lemma 5.3: If $\beta: U \to \mathbb{R}^r$ denotes a differentiable continuous function in $U \subseteq \mathbb{R}^r$ (a convex set) and if Π (a constant positive number less than one) exists such that $\mathbb{R}^r$ holds for any norm; $\| J_\beta(g) \| \leq \Pi, \forall \ g \in U$, then β has $\bar{g}$ (a unique fixed point) in U.

We can now confidently say that for any initial guess g^0 chosen in U, the iteration (4.3) converges to $\bar{g} \in \mathbb{R}^r$. To guarantee this uniqueness, let us obtain Jacobi Matrix (J_β).

Recall; $g := (p; q; r) := (p_1, \ldots, p_a\,; q_1, \ldots, q_b\,; r_1, \ldots, r_c) \in \mathbb{R}^{a+b+c}$

$$J_\beta = \frac{\alpha(\beta 1, \ldots, \beta a\,; \beta a+1, \ldots, \beta a+b\,; \beta a+b+1, \ldots, \beta a+b+c)}{\alpha(p1, \ldots, pa\,; q1, \ldots, qb\,; r1, \ldots, rc)}$$

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Where;

$$\frac{\alpha(\beta 1,\dots,\beta a)}{\alpha(q1,\dots,qb)} = \begin{pmatrix} 0 & \frac{\alpha\,\beta_1}{\alpha\,q_2} & \cdots & \frac{\alpha\,\beta_1}{\alpha\,q_b} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\alpha\,\beta_a}{\alpha\,q_2} & \cdots & \frac{\alpha\,\beta_a}{\alpha\,q_b} \end{pmatrix}$$

$$\frac{\alpha_{\beta_i}}{\alpha_{q_j}} = \frac{h_i^p \alpha(\sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{1jk})}{h_1^p\, \alpha_{q_j}(\sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{ijk})}$$

$$\frac{\alpha_{\beta_i}}{\alpha_{q_j}} = \frac{h_i^p}{h_1^p} \frac{(\sum_{k=1}^{c} r_k l_{1jk})\sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{ijk} - (\sum_{k=1}^{c} r_k l_{ijk})\sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{1jk}}{(\sum_{j=1}^{b} \sum_{k=1}^{c} q_j r_k l_{ijk})^2} \tag{5.4}$$

If $\frac{\alpha(\beta 1,\dots,\beta a)}{\alpha(p1,\dots,pa)} = \frac{\alpha(\beta a+1,\dots,\beta a+b)}{\alpha(q1,\dots,qb)} = \frac{\alpha(\beta a+b+1,\dots,\beta a+b+c)}{\alpha(r1,\dots,rc)} = 0$,

then the Jacobi Matrix (J_β) is represented by:

$$J_\beta = \begin{pmatrix} 0 & \frac{\alpha(\beta 1,\dots,\beta a)}{\alpha(q1,\dots,qb)} & \frac{\alpha(\beta 1,\dots,\beta a)}{\alpha(r1,\dots,rc)} \\ \frac{\alpha(\beta a+1,\dots,\beta a+b)}{\alpha(p1,\dots,pa)} & 0 & \frac{\alpha(\beta a+1,\dots,\beta a+b)}{\alpha(r1,\dots,rc)} \\ \frac{\alpha(\beta a+b+1,\dots,\beta a+b+c)}{\alpha(p1,\dots,pa)} & \frac{\alpha(\beta a+b+1,\dots,\beta a+b+c)}{\alpha(q1,\dots,qb)} & 0 \end{pmatrix} \tag{5.5}$$

Thus, if $0 \leq \frac{\alpha_{\beta_i}}{\alpha_{q_j}}$, minimising the denominator and maximising the numerator in (5.3), we obtain;

$$\frac{h_i^p (N_1^p \sum_{k=1}^{c} r_k)(N_i^p \sum_{j=1}^{b} q_j \sum_{k=1}^{c} r_k) - (\theta_1^p \sum_{k=1}^{c} r_k)(\theta_i^p \sum_{j=1}^{b} q_j \sum_{k=1}^{c} r_k)}{h_1^p(\theta_i^p \sum_{j=1}^{b} q_j \sum_{k=1}^{c} r_k)} \geq \frac{\alpha_{\beta_i}}{\alpha_{q_j}}$$

Alternatively, we get;

$$\frac{h_i^p(N_1^p N_i^p - \theta_1^p \theta_i^p)}{h_1^p(\theta_i^p)^2 \sum_{j=1}^{b} q_j} \geq \left|\frac{\alpha_{\beta_i}}{\alpha_{q_j}}\right|$$

It follows the same pattern if we apply the obtained result in the second and third inequalities in (5.3). Therefore, in every vector, $g=\{p_1,\dots,p_a; q_1,\dots,q_b; r_1,\dots,r_c\} \in U \subseteq \mathbb{R}^{a+b+c}$,

$$\Pi_\infty^i := \frac{h_i^p(N_1^p N_i^p - \theta_1^p \theta_i^p)}{h_1^p(\theta_i^p)^2} \left((b-1)\frac{h_1^q}{\theta_1^q \sum_{j=1}^{b} \frac{h_j^q}{N_j^q}} - (c-1)\frac{h_1^r}{\theta_1^r \sum_{k=1}^{c} \frac{h_k^r}{N_k^r}} \right) \geq$$

$$\sum_{j=1}^{b} \left|\frac{\alpha_{\beta_i}}{\alpha_{q_j}}\right| + \sum_{k=1}^{c} \left|\frac{\alpha_{\beta_i}}{\alpha_{r_k}}\right|$$

Similarly, Π_∞^j and Π_∞^k follow the same pattern where ;

$$\Pi_\infty := \max_{i,j,k}\{\Pi_\infty^i, \Pi_\infty^j, \Pi_\infty^k\}$$

Consequently, applying both (4.5) and (5.5), we derive;

$$\|J_\beta\|_\infty := \max_{i,j,k} \{\sum_{j=1}^{b} \left|\frac{\alpha\,\beta_i}{\alpha\,q_j}\right| + \sum_{k=1}^{c} \left|\frac{\alpha\,\beta_i}{\alpha\,r_k}\right|; \sum_{i=1}^{a} \left|\frac{\alpha\,\beta_{a+j}}{\alpha\,p_i}\right| + \sum_{k=1}^{c} \left|\frac{\alpha\,\beta_{a+j}}{\alpha\,r_k}\right|; \sum_{i=1}^{a} \left|\frac{\alpha\,\beta_{a+b+k}}{\alpha\,p_i}\right| + \sum_{j=1}^{b} \left|\frac{\alpha\,\beta_{a+b+k}}{\alpha\,q_j}\right|\}$$

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It is now crystal clear that $\Pi_\infty \geq \| J_\beta \|_\infty$. Therefore, function (5.2) possesses a unique fixed point and the algorithm (5.1) converges to the point as shown in Lemma (5.3) if $1 > \Pi_\infty$.

5.3 Sufficient Conditions for Algorithm Convergence

The next thing to do now is to examine the following theorems to prove the sufficient conditions for convergence of the iteration process of our algorithm in the formula (5.1)

Theorem 5.4: if $\Pi_\infty < 1$ and conditions of the Theorem (5.2) satisfied, the function defined by (5.2) will have a unique fixed point and for any initial guess g^0 in D^0, the algorithm (5.1) will converge towards that point.

Similarly, $\| J_\beta \|_1$ can also be obtained as;

$$\| J_\beta \|_1 = \max_{i,j,k} \left\{ \sum_{j=1}^{b} \left| \frac{\alpha\, \beta_{a+j}}{\alpha\, p_i} \right| + \sum_{k=1}^{c} \left| \frac{\alpha\, \beta_{a+b+k}}{\alpha\, p_i} \right| ; \quad \sum_{i=1}^{a} \left| \frac{\alpha\, \beta_i}{\alpha\, q_j} \right| + \sum_{k=1}^{c} \left| \frac{\alpha\, \beta_{a+b+k}}{\alpha\, q_j} \right| ; \quad \sum_{i=1}^{a} \left| \frac{\alpha\, \beta_i}{\alpha\, r_k} \right| + \sum_{j=1}^{b} \left| \frac{\alpha\, \beta_{a+j}}{\alpha\, r_k} \right| \right\}$$

Therefore,

$$\Pi_1^i := \frac{h_1^p}{\theta_1^p \sum_{i=1}^{a} \frac{h_1^p}{N_i^p}} \left\{ \sum_{j=1}^{b} \frac{h_j^q (N_1^q N_j^q - \theta_1^q \theta_j^q)}{h_1^q (\theta_j^q)^2} + \sum_{k=1}^{c} \frac{h_k^r (N_1^r N_k^r - \theta_1^r \theta_k^r)}{h_1^r (\theta_k^r)^2} \right\} \geq \sum_{j=1}^{b} \left| \frac{\alpha\, \beta_{a+j}}{\alpha\, p_i} \right| + \sum_{k=1}^{c} \left| \frac{\alpha\, \beta_{a+b+k}}{\alpha\, p_i} \right|, \ \forall \quad i=1,\ldots,m.$$

In this same way, Π_1^j and Π_1^k can be obtained. Thus;

$$\Pi_1 \quad := \quad \max_{i,j,k} \{ \Pi_1^i,\ \Pi_1^j,\ \Pi_1^k \}$$

The following statements follow as $\Pi_1 \geq \| J_\beta \|_1$:

Theorem 5.5: Assuming the conditions of Theorem 5.2 satisfied;

(1) The function defined by formula (5.2) has a unique fixed point and the algorithm 5.1 converges towards the point for any initial guess g^0 in D^0 if $1 > \Pi_1$

(2) The function β given by (5.2) also has a unique fixed point and the algorithm 5.1 converges towards the point for any initial guess g^0 in D^0 if $1 > \min\{\Pi_1, \Pi_\infty\} =: \Pi$

To significantly simplify these statements and derivations, we will introduce θ and N as ;

$$\theta \quad := \quad \min\{l_{ijk} : i = 1, \ldots, a ; j = 1, \ldots, b ; k = 1, \ldots, c \}$$

$$N \quad := \quad \max\{l_{ijk} : i = 1, \ldots, a ; j = 1, \ldots, b ; k = 1, \ldots, c \}$$

Then; $\frac{N h_1^q}{H\theta} \geq \frac{1}{\sum_{j=1}^{b} \frac{h_j^q \theta_1^q}{h_1^q N_j^q}} \geq \frac{1}{\sum_{k=1}^{b} q_k}$ and $\frac{h_i^p h_1^q N(N^2 - \theta^2)}{h_1^p \theta^3 H} \geq \frac{\alpha\, \beta_i}{\alpha\, q_j}$

5.4 Conclusion

By repeating the proof of Theorem 5.4 assuming the conditions of Theorem 5.2 satisfied and

$$1 > \frac{-N(\theta^2 - N^2)}{\theta^3 H} \max_{i,j,k} \left\{ \frac{(b-1) h_i^p h_1^q + (c-1) h_i^p h_1^r}{h_1^p} ; \frac{(a-1) h_j^q h_1^p + (c-1) h_j^q h_1^r}{h_1^q} ; \frac{(a-1) h_k^r h_1^p + (b-1) h_k^r h_1^q}{h_1^r} \right\} =: d_\infty ,$$

the function defined by formula 5.2 and the algorithm 5.1 have a unique fixed point and converge towards that point for any initial guess g^0 in D^0. By also repeating the proof of Theorem 5.5 with N and θ assuming the conditions of Theorem 5.2 satisfied and

$$1 > \frac{-N(\theta^2 - N^2)}{\theta^3} \, max\left\{ \frac{h_1^p}{h_1^q} + \frac{h_1^p}{h_1^r}; \quad \frac{h_1^q}{h_1^p} + \frac{h_1^q}{h_1^r}; \quad \frac{h_1^r}{h_1^p} + \frac{h_1^r}{h_1^q} \right\} =: d_1,$$

then the function defined by formula (5.2) has a unique fixed point and the algorithm 5.1 converges towards the point for any initial guess g^0 in D^0.

Consequently, if $1 > min\{d_1, d_\infty\}$=:d, the function β by (5.2) has a unique fixed point while the algorithm in (5.1) converges towards that point for any initial guess in g^0 chosen in D^0.

Note that the Class i for M > 3 risk classification variables shown in (4.10) and (4.11) will be denoted as;

$$h_i^p \coloneqq \sum_{j=1}^{b} \cdots \sum_{k=1}^{c} h_{ij\ldots k}, \qquad i = 1,2,\ldots,a \qquad (5.6)$$

$$L_i^p \coloneqq \sum_{j=1}^{b} \cdots \sum_{k=1}^{c} l_{ijk}\, q_j \ldots r_k, \quad i = 1,2,\ldots,a \qquad (5.7)$$

Both (5.6) and (5.7) pointed to the fact that ommission only occurs to summation by i^{th} index corresponding to factor p. Same can be said for other risk factors ($q=(q_1, \ldots, q_b),\ldots,r=(r_1,\ldots,r_c)$). That is to say, summation by the corresponding indices disappear as indicated in (5.6) and (5.7).

In the same vein, we can also generalise functions ($\beta_i, \ldots, \beta_{a+j}, \ldots, \beta_{a+b+\ldots+k}$) indicated in (5.2) as;

$$\beta_i = \frac{h_i^p \sum_{j=1}^{b} \cdots \sum_{k=1}^{c} l_{1jk} \ldots q_j \ldots r_k}{h_1^p \sum_{j=1}^{b} \cdots \sum_{k=1}^{c} l_{ijk}\, q_j \ldots r_k}, \qquad i=1,2,\ldots,a$$

$$\beta_{a+j} = \frac{h_j^q \sum_{i=1}^{a} \cdots \sum_{k=1}^{c} l_{i1k} \ldots p_i \ldots r_k}{h_1^q \sum_{i=1}^{a} \cdots \sum_{k=1}^{c} l_{ijk} \ldots p_i \ldots r_k}, \qquad j = 1,2,\ldots,b$$

$$\beta_{a+b+k} = \frac{h_k^r \sum_{i=1}^{a} \cdots \sum_{j=1}^{b} l_{ij1} \ldots p_i \ldots q_j}{h_1^r \sum_{j=1}^{b} \cdots \sum_{k=1}^{c} l_{ijk} \ldots p_i \ldots q_j}, \qquad k = 1,2,\ldots,c$$

Finally, all other formulae can be generalised in this same manner in order to make them straightforward and simplified.

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